Finite-size effects in a non-half-filled Hubbard chain

1989 J. Phys. A: Math. Gen. 22 4243

(http://iopscience.iop.org/0305-4470/22/19/017)

J. Phys. A: Math. Gen. 22 (1989) 4243-4256. Printed in the UK

Finite-size effects in a non-half-filled Hubbard chain

F Woynarovich
Central Research Institute for Physics, POB 49, H-1525 Budapest 114, Hungary

Received 14 February 1989

Abstract. The finite-size effects in the spectrum of a Hubbard chain are obtained for both the repulsive and attractive cases. It is shown that the finite-size corrections—similar to the case of a Heisenberg chain or Bose gas—are non-analytic unless some conditions are imposed on the chemical potential, magnetic field and chain length. If these conditions are met, the spectrum shows a similar tower structure as expected in conformal theories, although the model in general is not conformally invariant. In the special case when the two Fermi velocities are equal, the model is conformally invariant with $c=2$, the indices are similar to the Gaussian form and there are four marginal operators.

1. Introduction

In the understanding of the critical two-dimensional classical and $(1+1)$-dimensional quantum systems the concept of conformal symmetry put forward by Belavin *et al* (1984) has proven to be a very fruitful one. This symmetry provides an abstract classification according to the central charge (c) of the Virasoro algebra describing the conformal symmetry of the system (Friedan *et al* 1984). The conformal anomaly c and the scaling dimensions of the primary-order parameters are directly accessible through the finite-size effects in an affiliated system defined on an infinitely long but finitely wide strip (Blöte *et al* 1986, Affleck 1986). These results have prompted several groups to study the finite-size effects both numerically and analytically in different critical and conformally invariant systems.

A condition for a critical system to be also conformally invariant is that the group velocity be the same for all elementary excitations. If this holds, the spectrum of the Hamiltonian for a chain of length N (in 2D statistical systems, the spectrum of the logarithm of the transfer matrix acting along the infinitely long strip of width N) should have the so-called tower structure (Cardy 1986a, b) which in the most general case (Bogoliubov *et al* 1988, Berkovich and Murthy 1988) means

$$E(n, N^+, N^-) - E_0 = \frac{2\pi v_F}{N}(x_n + N^+ + N^-) \tag{1.1}$$

$$E_0 = N\varepsilon_{\rm inf} - \frac{\pi v_F}{6N} c \tag{1.2}$$

$$P(n, N^+, N^-) - P_0 = \frac{2\pi}{N}(s_n + N^+ + N^-) + D2k_F. \tag{1.3}$$

0305-4470/89/194243+14$02.50

Here $\varepsilon_{\rm inf}$ is the ground-state energy density of the infinite system and E_0 is the ground-state energy of the finite one, x_n and s_n are the scaling dimensions and spins of the primary scaling operators, N^+ and N^- are non-negative integers, D is the number of particles moved from one Fermi point to the other, P is the momentum of the system, $v_{\rm F}$ is the Fermi velocity and c is the central charge (conformal anomaly).

There are several examples for systems which are critical, but do not show conformal symmetry. These are systems in which there are several kinds of excitations, all posessing linear dispersion, but the different excitations have different velocities. Prime examples for such systems are the 1D spin-$\frac{1}{2}$ Fermi gas with δ interaction or its lattice version, the Hubbard chain. In these systems there are two kinds of excitations, one connected with the charge degrees of freedom, the other with the spins. Both are fermion-like but they have different Fermi velocities. These systems are not conformally invariant but are expected to be treatable in terms of two conformal fields (Korepin *et al* 1988).

In the present work we study the spectrum of the Hubbard chain. It is known that in the half-filled case the charge excitations possess a gap (Woynarovich 1982a, b, 1983a, b), only the spin excitations are critical and they have a spectrum of the form (1.1)-(1.3) with $c = 1$ and x_n, s_n of the Gaussian form (Woynarovich and Eckle 1987b). Now, to have both degrees of freedom critical, we study the non-half-filled band, and to have the possible most general case we introduce also a magnetic field. Thus the Hamiltonian is

$$\hat{H} = -\sum_{i=1}^{N}\sum_{\sigma}(c^{+}_{i+1,\sigma}c_{i,\sigma} + c^{+}_{i,\sigma}c_{i+1,\sigma}) + U\sum_{i=1}^{N} n_{i\uparrow}n_{i\downarrow}$$
$$+\mu\sum_{i=1}^{N}(n_{i\uparrow}+n_{i\downarrow}) - \tfrac{1}{2}h\sum_{i=1}^{N}(n_{i\uparrow}-n_{i\downarrow}) \tag{1.4}$$

where $c_{i,\sigma}$ ($\sigma = \downarrow$ or $\uparrow$) are the spin-$\frac{1}{2}$ fermion operators at site i, $n_{i\uparrow}$ and $n_{i\downarrow}$ are the numbers of up and down spin particles at site i, and μ and h are the chemical potential and magnetic field, respectively.

As is well known, this system is exactly treatable by the Bethe ansatz (Lieb and Wu 1968), and by now there is also a well established method to calculate the finite-size corrections in Bethe ansatz systems (de Vega and Woynarovich 1985, Woynarovich and Eckle 1987a). Treating the model this way we have determined the low-lying part (scaling with $1/N$) of the spectrum for the Hamiltonian (1.4) for both positive and negative U. We have found the following.

(i) This spectrum ((2.44) for $U > 0$ and (2.50) for $U < 0$) is not analytic in N unless extra conditions imposed on μ and h are also satisfied (similar to the case of the Heisenberg chain in magnetic field or the 1D Bose gas (Woynarovich *et al* 1989)).

(ii) If the extra conditions are satisfied, the spectrum has a structure (3.7) which can be considered as the generalisation of (1.2) and (1.3): it looks as the finite-size corrections would come from two independent $c = 1$ fields, but this independence is not true, however, since both of the x depend on the state of both Fermi seas.

(iii) There are special values of U, μ and h where the two Fermi velocities coincide. In these points the system is conformally invariant with $c = 2$. The scaling indices are of generalised Gaussian form (3.21) (similar to that found in nested Bethe ansatz systems by Suzuki (1988)), and there are four marginal operators.

In the next section we give a detailed derivation of the energy and in § 3 we discuss our results in detail.

2. The energy of a finite chain

The Bethe ansatz equations for the Hubbard chain are

$$Nk_j = 2\pi I_j + \sum_{\beta=1}^{N_s} 2\tan^{-1}\left(\frac{\sin k_j - \lambda_\beta}{u}\right) \tag{2.1a}$$

$$\sum_{j=1}^{N_c} 2\tan^{-1}\left(\frac{\lambda_\alpha - \sin k_j}{u}\right) = 2\pi J_\alpha + \sum_{\beta=1}^{N_s} 2\tan^{-1}\left(\frac{\lambda_\alpha - \lambda_\beta}{2u}\right). \tag{2.1b}$$

Here N_c is the total number of particles and N_s is the number of down spins, u is the interaction strength in units of the bandwidth $u = U/4$, and the quantum numbers I_j and J_α are integers or half-odd-integers, depending on the parities of the numbers N_c and N_s:

$$I_j = N_s/2 \,(\mathrm{mod}\, 1) \qquad J_\alpha = (N_c + N_s + 1)/2 \,(\mathrm{mod}\, 1). \tag{2.2}$$

Once these equations are solved, the energy and the momentum of the system are given by

$$E = -2\sum_{j=1}^{N_c} \cos k_j + \mu N_c + h(N_s - N_c/2) \tag{2.3}$$

$$P = \sum_{j=1}^{N_c} k_j = \frac{2\pi}{N}\left(\sum_j I_j + \sum_\alpha J_\alpha\right). \tag{2.4}$$

We solve these equations at $U > 0$ for those states which have a spectrum scaling like $1/N$. (The $U < 0$ case can be obtained from the $U > 0$ one through a transformation (Woynarovich 1983b).) For this we choose the I_j and J_α sets as follows: we choose $I^\pm = (N_s + 1)/2$ (mod 1) and $J^\pm = (N_c + N_s)/2$ (mod 1) so that

$$\begin{aligned} I^+ - I^- &= N_c \qquad & I^+ + I^- &= 2D_c \\ J^+ - J^- &= N_s \qquad & J^+ + J^- &= 2D_s \end{aligned} \tag{2.5}$$

with $D_{c(s)} \ll N$. The I_j are all the numbers equal to $N_s/2$ (mod 1) between I^+ and I^- while the J_α are all the numbers equal to $(N_c + N_s + 1)/2$ (mod 1) between J^+ and J^-. This corresponds to two Fermi seas with D_c and D_s particles moved from the left Fermi points to the right ones. (Later on particle-hole pairs can also be introduced but care must be taken that the number of holes and particles must be the same around all four Fermi points separately (i.e. not to change N_c, N_s, D_c and D_s). Excitations with complex k and λ will not be considered as they have a gap.)

Now we define

$$\begin{aligned} z_c(k) &= \frac{1}{2\pi}\left[k + \frac{1}{N}\sum_\beta 2\tan^{-1}\left(\frac{\sin k - \lambda_\beta}{u}\right)\right] \\ z_s(\lambda) &= \frac{1}{2\pi}\left[\frac{1}{N}\sum_j 2\tan^{-1}\left(\frac{\lambda\text{-}\sin k_j}{u}\right) - \frac{1}{N}\sum_\beta 2\tan^{-1}\left(\frac{\lambda - \lambda_\beta}{2u}\right)\right] \end{aligned} \tag{2.6}$$

and

$$\rho_c(k) = \partial z_c(k)/\partial k \qquad \rho_s(\lambda) = \partial z_s(\lambda)/\partial\lambda. \tag{2.7}$$

With this notation

$$z_c(k_j) = I_j/N \qquad z_s(\lambda_\alpha) = J_\alpha/N. \tag{2.8}$$

Using the formula

$$\frac{1}{N}\sum_{n=n_1}^{n_2} f\left(\frac{n}{N}\right) \simeq \int_{(n_1-1/2)/N}^{(n_2+1/2)/N} f(x)\,\mathrm{d}x + \frac{1}{24N^2}\left(f'\left(\frac{n_1-1/2}{N}\right) - f'\left(\frac{n_2+1/2}{N}\right)\right) \tag{2.9}$$

(2.7) can be written in the form

$$\rho_c(k) = \frac{1}{2\pi}\left(1 - \frac{1}{24N^2\rho_s(\lambda^-)}\cos(k)K_1'(\sin k - \lambda^-) + \frac{1}{24N^2\rho_s(\lambda^+)}\cos(k)K_1'(\sin k - \lambda^+)\right.$$
$$\left. + \cos(k)\int_{\lambda^-}^{\lambda^+} K_1(\sin k - \lambda')\rho_s(\lambda')\,\mathrm{d}\lambda'\right) \tag{2.10a}$$

$$\rho_s(\lambda) = \frac{1}{2\pi}\left(-\frac{1}{24N^2\rho_c(k^-)}\cos(k^-)K_1'(\lambda - \sin k^-)\right.$$
$$+\frac{1}{24N^2\rho_c(k^+)}\cos(k^+)K_1'(\lambda - \sin k^+)$$
$$+\frac{1}{24N^2\rho_s(\lambda^-)}K_2'(\lambda-\lambda^-) - \frac{1}{24N^2\rho_s(\lambda^+)}K_2'(\lambda-\lambda^+)$$
$$\left.+\int_{k^-}^{k^+} K_1(\lambda - \sin k')\rho_c(k')\,\mathrm{d}k' - \int_{\lambda^-}^{\lambda^+} K_2(\lambda-\lambda')\rho_s(\lambda')\,\mathrm{d}\lambda'\right). \tag{2.10b}$$

Here

$$K_1(x) = 2\frac{u}{u^2+x^2} \qquad K_2(x) = 2\frac{2u}{(2u)^2+x^2}. \tag{2.11}$$

and $K_{1,2}'$ are the derivatives of $K_{1,2}$ and $k^{\pm}$ and $\lambda^{\pm}$ satisfy the equations

$$z_c(k^{\pm}) = I^{\pm}/N \qquad z_s(\lambda^{\pm}) = J^{\pm}/N. \tag{2.12}$$

These four equations (2.12), together with the definitions of ρ and (2.5), are equivalent to

$$\int_{k^+}^{k^+} \rho_c(k) = \frac{N_c}{N} \qquad -\frac{1}{2}\left(\int_{k^+}^{\pi}\rho_c(k) - \int_{-\pi}^{k^-}\rho_c(k)\right) - \frac{1}{2\pi}\int_{\lambda^-}^{\lambda^+} 2\tan^{-1}(\lambda/u)\rho_s(\lambda) = \frac{D_c}{N} \tag{2.13}$$

$$\int_{\lambda^-}^{\lambda^+}\rho_s(\lambda) = \frac{N_s}{N} \qquad -\frac{1}{2}\left(\int_{\lambda^+}^{\infty}\rho_s(\lambda) - \int_{-\infty}^{\lambda^-}\rho_s(\lambda)\right) = \frac{D_s}{N}. \tag{2.14}$$

In the following a central role will be played by the solutions of equations of the type

$$x_c(k\,|\,k^{\pm},\lambda^{\pm}) = x_{0c}(k) + \frac{\cos k}{2\pi}\int_{\lambda^-}^{\lambda^+} K_1(\sin k - \lambda')x_s(\lambda'|\,k^{\pm},\lambda^{\pm})\,\mathrm{d}\lambda'$$
$$x_s(\lambda\,|\,k^{\pm},\lambda^{\pm}) = x_{0s}(\lambda) + \frac{1}{2\pi}\int_{k^-}^{k^+} K_1(\lambda - \sin k')x_c(k'|\,k^{\pm},\lambda^{\pm})\,\mathrm{d}k' \tag{2.15}$$
$$-\frac{1}{2\pi}\int_{\lambda^-}^{\lambda^+} K_2(\lambda-\lambda')x_s(\lambda'|\,k^{\pm},\lambda^{\pm})\,\mathrm{d}\lambda'.$$

This system of equations we shall write in the symbolic form

$$\boldsymbol{x}(k,\lambda\,|\,k^{\pm},\lambda^{\pm})=\boldsymbol{x}_0(k,\lambda)+\mathbf{K}(k,\lambda\,|\,k',\lambda'|\,k^{\pm},\lambda^{\pm})\otimes\boldsymbol{x}(k',\lambda'|\,k^{\pm},\lambda^{\pm}). \tag{2.16}$$

Here $\boldsymbol{x}(k,\lambda)$ is a column vector with upper and lower elements $x_c(k)$ and $x_s(\lambda)$, respectively, and $\mathbf{K}$ is a 2×2 matrix with integral-operator elements which can be read out from (2.15). We shall also use the equation

$$\boldsymbol{y}(k,\lambda\,|\,k^{\pm},\lambda^{\pm})=\boldsymbol{y}_0(k,\lambda)+\mathbf{K}^{\mathrm{T}}(k,\lambda\,|\,k',\lambda'|\,k^{\pm},\lambda^{\pm})\otimes\boldsymbol{y}(k',\lambda'|\,k^{\pm},\lambda^{\pm}) \tag{2.17}$$

which is analogous to (2.15) but the integral-operator matrix $\mathbf{K}^{\mathrm{T}}$ is the transpose of that in (2.15):

$$\mathbf{K}^{\mathrm{T}}(k,\lambda\,|\,k',\lambda'|\,k^{\pm},\lambda^{\pm}) = \frac{1}{2\pi}\begin{pmatrix} 0 & \int_{\lambda^-}^{\lambda^+}\mathrm{d}\lambda'\,K_1(\sin k-\lambda')\ldots \\ \int_{k^-}^{k^+}\mathrm{d}k'\,K_1(\lambda-\sin k')\cos k'\ldots & \int_{\lambda^-}^{\lambda^+}\mathrm{d}\lambda'\,K_2(\lambda-\lambda')\ldots \end{pmatrix}. \tag{2.18}$$

It is clear that

$$\begin{aligned}\boldsymbol{\rho}(k,\lambda)=\boldsymbol{\rho}_\infty(k,\lambda)+\frac{1}{24N^2}\bigg(&\frac{1}{\rho_c(k^+)}\boldsymbol{\rho}_1(k,\lambda\,|\,k^{\pm},\lambda^{\pm})\\ &+\frac{1}{\rho_c(k^-)}\boldsymbol{\rho}_1(-k,-\lambda\,|\,-k^{\mp},-\lambda^{\mp})+\frac{1}{\rho_s(\lambda^+)}\boldsymbol{\rho}_2(k,\lambda\,|\,k^{\pm},\lambda^{\pm})\\ &+\frac{1}{\rho_s(\lambda^-)}\boldsymbol{\rho}_2(-k,-\lambda\,|\,-k^{\mp},-\lambda^{\mp})\bigg)\end{aligned} \tag{2.19}$$

with $\boldsymbol{\rho}_\infty$, $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ determined by (2.16) with the inhomogeneous part $\boldsymbol{x}_0$ replaced by

$$\begin{pmatrix}\frac{1}{2\pi}\\ 0\end{pmatrix}\qquad\begin{pmatrix}0\\ \frac{1}{2\pi}\cos(k^+)K_1'(\lambda-\sin k^+)\end{pmatrix}\qquad\begin{pmatrix}\frac{1}{2\pi}\cos(k)K_1'(\sin k-\lambda^+)\\ -\frac{1}{2\pi}K_2'(\lambda-\lambda^+)\end{pmatrix} \tag{2.20}$$

respectively.

The energy according to (2.3) and (2.9) is

$$\begin{aligned}E=N\int_{k^-}^{k^+}(h_c-2\cos k)\rho_c(k)+Nh_s\int_{\lambda^-}^{\lambda^+}\rho_s(\lambda)\\ +\frac{1}{24N\rho_c(k^+)}2\sin k^+-\frac{1}{24N\rho_c(k^-)}2\sin k^-\end{aligned} \tag{2.21}$$

with $h_c=\mu-h/2$ and $h_s=h$. This, using (2.19), can be written in the form

$$\begin{aligned}E=N\varepsilon_\infty(k^+,k^-,\lambda^+,\lambda^-)+\frac{1}{24N}(&\varepsilon_1(k^+,k^-,\lambda^+,\lambda^-)\\ &+\varepsilon_1(-k^-,-k^+,-\lambda^-,-\lambda^+)+\varepsilon_2(k^+,k^-,\lambda^+,\lambda^-)\\ &+\varepsilon_2(-k^-,-k^+,-\lambda^-,-\lambda^+))\end{aligned} \tag{2.22}$$

with

$$\varepsilon_\infty = \int_{k^-}^{k^+} (h_c - 2\cos k)\rho_{\infty_c}(k) + \int_{\lambda^-}^{\lambda^+} h_s \rho_{\infty_s}(\lambda) \tag{2.23}$$

$$\varepsilon_1 = \frac{1}{\rho_c(k^+)}\left(2\sin k^+ - \int_{k^-}^{k^+} (h_c - 2\cos k)\rho_{1c}(k) - \int_{\lambda^-}^{\lambda^+} h_s \rho_{1s}(\lambda)\right) \tag{2.24}$$

$$\varepsilon_2 = \frac{-1}{\rho_s(\lambda^+)}\left(\left(\int_{k^-}^{k^+} (h_c - 2\cos k)\rho_{2c}(k) + \int_{\lambda^-}^{\lambda^+} h_s \rho_{2s}(\lambda)\right)\right). \tag{2.25}$$

ε_∞ is actually the energy density of an infinite system at the given $k^\pm$ and $\lambda^\pm$. It is instructive to write it in another form. Since the formal solution of an equation of the type (2.16) is

$$\boldsymbol{x}(k,\lambda) = \sum_{n=1}^{\infty} (\mathbf{K}(k,\lambda\,|\,k',\lambda'\,|\,k^\pm,\lambda^\pm)\otimes)^n \boldsymbol{x}_0(k',\lambda') \tag{2.26}$$

ρ_∞ will be given by (2.26) with $\boldsymbol{x}_0$ replaced by the first column vector in (2.20). Substituting this into (2.23) one obtains that

$$\varepsilon_\infty = \frac{1}{2\pi}\int_{k^-}^{k^+} \varepsilon_c(k\,|\,k^\pm,\lambda^\pm)\,\mathrm{d}k \tag{2.27}$$

where

$$\boldsymbol{\varepsilon}(k,\lambda\,|\,k^\pm,\lambda^\pm) = \sum_{n=1}^{\infty} (\mathbf{K}^{\mathrm{T}}(k,\lambda\,|\,k',\lambda'\,|\,k^\pm,\lambda^\pm)\otimes)^n \boldsymbol{\varepsilon}_0(k',\lambda') \tag{2.28}$$

with

$$\boldsymbol{\varepsilon}_0 = \begin{pmatrix} h_c - 2\cos k \\ h_s \end{pmatrix} \tag{2.29}$$

i.e. $\boldsymbol{\varepsilon}(k,\lambda)$ satisfies (2.17) with $\boldsymbol{y}_0$ replaced by $\boldsymbol{\varepsilon}_0$. $\varepsilon_c(k)$ and $\varepsilon_s(\lambda)$ can be considered as the dressed energies.

The infinite chain is in the ground state at the given μ and h if ε_∞ is minimal with respect to $k^\pm$ and $\lambda^\pm$. This condition, using (2.27) and the integral equations determining ε_c, lead to the conditions

$$\varepsilon_c(k^+|\,k^\pm,\lambda^\pm) = 0 \qquad \varepsilon_c(k^-|k^\pm,\lambda^\pm) = 0 \tag{2.30}$$

$$\varepsilon_s(\lambda^+|\,k^\pm,\lambda^\pm) = 0 \qquad \varepsilon_s(\lambda^-|\,k^\pm,\lambda^\pm) = 0 \tag{2.31}$$

i.e. in the ground state the dressed energies are zero at the Fermi points. From symmetry it is clear that in the ground state $k^- = -k^+$ and $\lambda^- = -\lambda^+$. Let us denote the ground-state values of k^+ and λ^+ by k_0 and λ_0, respectively. Now we can expand ε_∞ up to second order in $(k^\pm \mp k_0)$ and $(\lambda^\pm \mp \lambda_0)$. Since the conditions (2.30) and (2.31) are satisfied at $k^\pm = \pm k_0$, $\lambda^\pm = \pm\lambda_0$ there are no cross derivatives and we find

$$\begin{aligned}\varepsilon_\infty(k^+,k^-,\lambda^+,\lambda^-) &= \varepsilon_\infty(k_0,-k_0,\lambda_0,-\lambda_0) + \frac{1}{\rho_{\infty c}(k_0)}\frac{\partial}{\partial k}\,\varepsilon c(k\,|\pm k_0,\pm\lambda_0)\Big|_{k=k_0}\\ &\quad\times\tfrac{1}{2}[(\rho_{\infty c}(k_0)(k^+ - k^0))^2 + (\rho_{\infty c}(k_0)(k^- + k^0))^2]\\ &\quad+\frac{1}{\rho_{\infty s}(\lambda_0)}\frac{\partial}{\partial\lambda}\,\varepsilon_s(\lambda\,|\pm k_0,\pm\lambda_0)\Big|_{\lambda=\lambda_0}\\ &\quad\times\tfrac{1}{2}[(\rho_{\infty s}(\lambda_0)(\lambda^+ - \lambda_0))^2 + (\rho_{\infty s}(\lambda_0)(\lambda^- + \lambda_0))^2].\end{aligned} \tag{2.32}$$

Utilising the equation for $\boldsymbol{\varepsilon}$ and those for $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ it is not hard to see that with an accuracy of $1/N^2$ (the error coming from the $O(1/N^2)$ difference between $\boldsymbol{\rho}$ and $\boldsymbol{\rho}_\infty$)

$$\begin{aligned}\frac{1}{\rho_{\infty c}(k_0)}\frac{\partial}{\partial k}\varepsilon_c(k\,|\pm k_0,\pm\lambda_0)\bigg|_{k=k_0} &= \varepsilon_1(k_0,-k_0,\lambda_0,-\lambda_0)\\ \frac{1}{\rho_{\infty s}(\lambda_0)}\frac{\partial}{\partial\lambda}\varepsilon_s(\lambda\,|\pm k_0,\pm\lambda_0)\bigg|_{\lambda=\lambda_0} &= \varepsilon_2(k_0,-k_0,\lambda_0,-\lambda_0).\end{aligned} \tag{2.33}$$

As a next step we express $(k^\pm \mp k_0)$ and $(\lambda^\pm \mp \lambda_0)$ by the deviation of N_c, N_s, D_c and D_s from their ground-state values. For this we can use (2.13a, b). First we note that, since the energy contains the square of these deviations only, it is enough to calculate them up to $O(1/N)$, i.e. in (2.13a, b) ρ_c and ρ_s can be replaced by $\rho_{\infty c}$ and $\rho_{\infty s}$, respectively. Denoting $N_c/N=\nu_c$, $N_s/n=\nu_s$, $D_c/N=\delta_c$, $D_s/N=\delta_s$, from (2.13a, b) we have

$$\frac{\partial\nu_c}{\partial k^+} = -\frac{\partial\nu_c}{\partial k^-} = \rho_{\infty c}(k_0)\left(1+\int_{-k_0}^{k_0}\sigma_{1c}(k)\right) = \rho_{\infty c}(k_0)\xi_{11} \tag{2.34a}$$

$$\frac{\partial\nu_s}{\partial k^+} = -\frac{\partial\nu_s}{\partial k^-} = \rho_{\infty c}(k_0)\int_{-\lambda_0}^{\lambda_0}\sigma_{1s}(\lambda) = \rho_{\infty c}(k_0)\xi_{12} \tag{2.34b}$$

$$\frac{\partial\nu_c}{\partial \lambda^+} = -\frac{\partial\nu_c}{\partial \lambda^-} = \rho_{\infty s}(\lambda_0)\int_{-k_0}^{k_0}\sigma_{2c}(k) = \rho_{\infty s}(\lambda_0)\xi_{21} \tag{2.34c}$$

$$\frac{\partial\nu_s}{\partial \lambda^+} = -\frac{\partial\nu_s}{\partial \lambda^-} = \rho_{\infty s}(\lambda_0)\left(1+\int_{-\lambda_0}^{\lambda_0}\sigma_{2s}(\lambda)\right) = \rho_{\infty s}(\lambda_0)\xi_{22} \tag{2.34d}$$

$$\begin{aligned}\frac{\partial\delta_c}{\partial k^+} = \frac{\partial\delta_c}{\partial k^-} = \rho_{\infty c}(k_0)\bigg[\frac{1}{2}\bigg(1-\int_{k_0}^{\pi}\sigma_{1c}+\int_{-\pi}^{-k_0}\sigma_{1c}\bigg)&\\ -\frac{1}{\pi}\int_{-\lambda_0}^{\lambda_0}\tan^{-1}(\lambda/u)\sigma_{1s}\bigg] = \rho_{\infty c}(k_0)z_{11}&\end{aligned} \tag{2.35a}$$

$$\frac{\partial\delta_s}{\partial k^+} = \frac{\partial\delta_s}{\partial k^-} = -\rho_{\infty c}(k_0)\frac{1}{2}\left(\int_{\lambda_0}^{\infty}\sigma_{1s}-\int_{-\infty}^{-\lambda_0}\sigma_{1s}\right) = \rho_{\infty c}(k_0)z_{12} \tag{2.35b}$$

$$\begin{aligned}\frac{\partial\delta_c}{\partial \lambda^+} = \frac{\partial\delta_c}{\partial \lambda^-} = -\rho_{\infty s}(\lambda_0)\bigg[\frac{1}{2}\bigg(\int_{k_0}^{\pi}\sigma_{2c}-\int_{-\pi}^{-k_0}\sigma_{2c}\bigg)&\\ +\frac{1}{\pi}\tan^{-1}(\lambda_0/u)+\frac{1}{\pi}\int_{-\lambda_0}^{\lambda_0}\tan^{-1}(\lambda/u)\sigma_{2s}\bigg] = \rho_{\infty_s}(\lambda_0)z_{21}&\end{aligned} \tag{2.35c}$$

$$\frac{\partial\delta_s}{\partial \lambda^+} = \frac{\partial\delta_s}{\partial \lambda^-} = \rho_{\infty s}(\lambda_0)\frac{1}{2}\left(1-\int_{\lambda_0}^{\infty}\sigma_{2s}+\int_{-\infty}^{-\lambda_0}\sigma_{2s}\right) = \rho_{\infty s}(\lambda_0)z_{22} \tag{2.35d}$$

where $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are defined by (2.16) with the inhomogeneous part $\boldsymbol{x}_0$ replaced by

$$\begin{pmatrix} 0 \\ \dfrac{1}{2\pi}K_1(\lambda-\sin k_0)\end{pmatrix} \qquad \begin{pmatrix}\dfrac{1}{2\pi}\cos(k)K_1(\sin k-\lambda_0)\\ -\dfrac{1}{2\pi}K_2(\lambda-\lambda_0)\end{pmatrix} \tag{2.36}$$

respectively. Using the formal solution (2.26) of (2.16) for $\boldsymbol{\sigma}_1$ one can convince oneself that the matrix $\boldsymbol{\xi}$ in (2.34) is

$$\boldsymbol{\xi}=\begin{pmatrix}\xi_{11} & \xi_{12}\\ \xi_{21} & \xi_{22}\end{pmatrix}=\begin{pmatrix}\xi_{11}(k_0) & \xi_{12}(k_0)\\ \xi_{21}(\lambda_0) & \xi_{22}(\lambda_0)\end{pmatrix} \tag{2.37}$$

where the $\boldsymbol{\xi}(k,\lambda)$ matrix is defined through the equation

$$\boldsymbol{\xi}(k,\lambda)=\mathbf{I}+\mathbf{K}^{\mathrm{T}}(k,\lambda\,|\,k',\lambda'|\pm k_0,\pm\lambda_0)\otimes\boldsymbol{\xi}(k',\lambda') \tag{2.38}$$

with $\mathbf{I}$ being the 2×2 identity matrix. This $\boldsymbol{\xi}$ matrix can be considered as a generalisation of the dressed charge (Korepin 1979). Taking the derivatives of the elements of $\boldsymbol{\xi}$ and reintegrating the first row from k_0 to π and the second row from λ_0 to infinity one finds that the $\mathbf{z}$ matrix in (2.35) is

$$\mathbf{z}=\tfrac{1}{2}(\boldsymbol{\xi}^{\mathrm{T}})^{-1} \tag{2.39}$$

with the upper index T corresponding to transposition. Compiling (2.22), (2.32), (2.33)-(2.37) and (2.39) one arrives at the energy expression correct to $\mathrm{O}(1/N^2)$

$$\begin{aligned}E=N\varepsilon_\infty(k_0,-k_0,\lambda_0,-\lambda_0)&\\ +\frac{1}{N}\varepsilon_1&\left(\frac{[\xi_{22}(N_c-\nu_cN)-\xi_{21}(N_s-\nu_sN)]^2}{4(\det\xi)^2}+(\xi_{11}D_c+\xi_{12}D_s)^2-\frac{1}{12}\right)\\ +\frac{1}{N}\varepsilon_2&\left(\frac{[\xi_{12}(N_c-\nu_cN)-\xi_{11}(N_s-\nu_sN)]^2}{4(\det\xi)^2}+(\xi_{21}D_c+\xi_{22}D_s)^2-\frac{1}{12}\right)\end{aligned} \tag{2.40}$$

where ε_1 and ε_2 are given by (2.24) and (2.25) with $k^{\pm}=\pm k_0$ and $\lambda^{\pm}=\pm\lambda_0$, and ν_c and ν_s are the densities of the particles and down spins, respectively, in the ground state of an infinite system at μ and h.

The total momentum of the system according to (2.4) and (2.5) is given by

$$P=\frac{1}{2\pi}(N_cD_c+N_sD_s). \tag{2.41}$$

As a final step we have to deal with the excitations. As an example we treat a particle-hole pair around the right Fermi point of the k sea. Suppose that the particle and hole are characterised by the quantum numbers I_{p}^+ and I_{h}^+, respectively. This defines their positions in the k space:

$$z_c(k_{\mathrm{p}}^+)=I_{\mathrm{p}}^+/N \qquad z_c(k_{\mathrm{h}}^+)=I_{\mathrm{h}}^+/N. \tag{2.42}$$

The presence of this particle-hole pair modifies $\boldsymbol{\rho}(k,\lambda)$ by $-\boldsymbol{\rho}_1(k,\lambda\,|\,k^{\pm},\lambda^{\pm})(k_{\mathrm{p}}^+-k_{\mathrm{h}}^+)/N$ and gives a contribution to the energy $\varepsilon_1\rho_c(k^+)(k_{\mathrm{p}}^+-k_{\mathrm{h}}^+)/N$. According to (2.42) and the definition of ρ, $\rho_c(k^+)(k_{\mathrm{p}}^+-k_{\mathrm{h}}^+)=(I_{\mathrm{p}}^+-I_{\mathrm{h}}^+)/N$. The momentum of such a particle-hole pair is $2\pi(I_{\mathrm{p}}^+-I_{\mathrm{h}}^+)/N$ and thus, due to this excitation we have contributions $\varepsilon_1(I_{\mathrm{p}}^+-I_{\mathrm{h}}^+)/N$ and $2\pi(I_{\mathrm{p}}^+-I_{\mathrm{h}}^+)/N$ to the energy (2.40) and momentum (2.41), respectively. This and the analogue calculations for the particle-hole excitations in the λ sea justify the notation

$$\varepsilon_1=2\pi v_c \qquad \varepsilon_2=2\pi v_s \tag{2.43}$$

with v_c and v_s being the Fermi velocities for the two Fermi seas. It is not hard to see that the energy and the momentum of a state with all four possible particle-hole excitations are

$$E = N\varepsilon_\infty(k_0, -k_0, \lambda_0, -\lambda_0) + \frac{2\pi v_c}{N}\left(\frac{[\xi_{22}(N_c-\nu_c N)-\xi_{21}(N_s-\nu_s N)]^2}{4(\det\xi)^2} + (\xi_{11}D_c+\xi_{12}D_s)^2 - \tfrac{1}{12} + n_c^+ + n_c^-\right) + \frac{2\pi v_s}{N}\left(\frac{[\xi_{12}(N_c-\nu_c N)-\xi_{11}(N_s-\nu_s N)]^2}{4(\det\xi)^2} + (\xi_{21}D_c+\xi_{22}D_s)^2 - \tfrac{1}{12} + n_s^+ + n_s^-\right) \tag{2.44}$$

$$P = \frac{2\pi}{N}(N_c D_c + n_c^+ - n_c^- + N_s D_s + n_s^+ - n_s^-). \tag{2.45}$$

Here

$$n_c^+ = \sum_{\mathrm{p}} I_{\mathrm{p}}^+ - \sum_{\mathrm{h}} I_{\mathrm{h}}^+ \qquad n_c^- = \sum_{\mathrm{h}} I_{\mathrm{h}}^- - \sum_{\mathrm{p}} I_{\mathrm{p}}^-$$
$$n_s^+ = \sum_{\mathrm{p}} J_{\mathrm{p}}^+ - \sum_{\mathrm{h}} J_{\mathrm{h}}^+ \qquad n_s^- = \sum_{\mathrm{h}} J_{\mathrm{h}}^- - \sum_{\mathrm{p}} J_{\mathrm{p}}^- \tag{2.46}$$

With I_{p}^+ (I_{h}^+), I_{p}^- (I_{h}^-), J_{p}^+ (J_{h}^+) and J_{p}^- (J_{h}^-) being the quantum numbers of the particles (holes) near to the right $(+)$ and left $(-)$ Fermi points of the k and λ seas.

The above results are valid for the case $U > 0$ but can be easily translated for negative U. A way to do this is provided by the 'complementer solutions' of the Lieb-Wu equations (2.1*a*, *b*) (Woynarovich 1983b). Suppose that in a $\{k_j, \lambda_\alpha\}$ solution of these equations the k_j are distributed in (k^+, k^-) according to $\rho_c(k)$ with holes at k_{h} and particles at k_{p}. In the complementer solution the λ_α are unchanged, but the k_j set is replaced by a k_g set in which there are real k and also complex k pairs. The real k_g are distributed in $(-\pi, k^-)$ and (k^+, π) according the corresponding part of $\rho_c(k)$ ($\rho_c(k)$ is defined by (2.7) for the whole $(-\pi, \pi)$ interval) with holes at k_{p} and particles at k_{h}, while the complex k_g pairs are determined by the λ:

$$\sin k_\alpha^\pm = \lambda_\alpha \mp \mathrm{i}u \qquad \pm \mathrm{Im}\, k_\alpha^\pm > 0. \tag{2.47}$$

The total number of k_g is $N + 2N_s - N_c$. Also this $\{k_g, \lambda_\alpha\}$ set will solve (2.1*a*, *b*) of course with an $\{I_g, J'_\alpha\}$ set different from $\{I_j, J_\alpha\}$ (but $I_j - I_g =$ integer and $J_\alpha - J'_\alpha =$ integer). The sum of the energies and momenta of these complementer solutions is

$$\sum -2\cos k_j + \sum -2\cos k_g = N_s U$$
$$\sum k_j + \sum k_g = \pi(N + N_s + 1). \tag{2.48}$$

Utilising all this one can see that the set $\{k_g + \pi, -\lambda_\alpha\}$ will satisfy (2.1*a*, *b*) with u replaced by $-u$ and $\{I_j, J_\alpha\}$ replaced by $\{I_g + N/2, J'_\alpha\}$. This solution describes an eigenstate of the attractive chain (interaction: $-U$) in which there are $N_{\mathrm{f}} = N - N_c$ 'free' particles and $N_b = N_s$ bound pairs, and the total spin is $S = N_{\mathrm{f}}/2$. (The parities of the numbers $2I_g + N$ and $2J'_\alpha$ will correspond to the periodic boundary condition (2.2), if in the positive u equations the quantum numbers are chosen as $I_j = (N + N_{\mathrm{b}})/2$ (mod 1); $J_\alpha = (N_{\mathrm{b}} + N_{\mathrm{f}} + 1)/2$ (mod 1) (accordingly $I^\pm = (N + N_{\mathrm{b}} + 1)/2$ (mod 1) and $J^\pm = (N_{\mathrm{f}} + N_{\mathrm{b}})/2$ (mod 1)).) Through (2.48) one can see that, if for the chain with $-U$

we chose $\mu'=(U+h)/2$ and $h'=U+2\mu$, the eigenstate for this attractive chain described by $\{k_g+\pi, -\lambda_\alpha\}$ will have the same energy as the state $\{k_j, \lambda_\alpha\}$ in the repulsive chain (1.4) (apart from a macroscopic constant). Thus, after substituting

$$\begin{aligned} &N_c = N - N_f \qquad D_c = D_f \qquad \nu_c = 1-\nu_f \qquad n_c^{\pm} = n_f^{\mp} \\ &N_s = N_b \qquad D_s = -D_b \qquad \nu_s = \nu_b \qquad n_s^{\pm} = n_b^{\mp} \end{aligned} \tag{2.49}$$

into (2.44) we have

$$\begin{aligned} E(-U) = {} & N(\varepsilon_\infty(k_0, -k_0, \lambda_0, -\lambda_0) - \mu + h/2) \\ & + \frac{2\pi v_c}{N}\bigg(\frac{[\xi_{22}(N_f - \nu_f N) + \xi_{21}(N_b - \nu_b N)]^2}{4(\det\xi)^2} \\ & + (\xi_{11}D_f - \xi_{12}D_b)^2 - \tfrac{1}{12} + n_f^+ + n_f^-\bigg) \\ & + \frac{2\pi v_s}{N}\bigg(\frac{[\xi_{12}(N_f - \nu_f N) + \xi_{11}(N_b - \nu_b N)]^2}{4(\det\xi)^2} \\ & + (\xi_{21}D_f - \xi_{22}D_b)^2 - \tfrac{1}{12} + n_b^+ + n_b^-\bigg) \end{aligned} \tag{2.50}$$

and (2.45), (2.48) and (2.49) gives

$$P = \pi N + \frac{2\pi}{N}(N_f D_f + n_f^+ - n_f^- + N_b D_b + n_b^+ - n_b^-). \tag{2.51}$$

Since $2\nu_s \leqslant \nu_c < 1$, $\nu_f + 2\nu_b \leqslant 1$.

3. Discussion of the energy spectrum and finite-size effects

First we have to notice that the spectrum (2.44) is not analytic in N: for the finite system in the ground state N_c and N_s should minimise (2.44), but since N_c, N_s and N are integers, the optimal values of N_c and N_s are not analytic functions of N. This phenomenon is known already for other systems (Woynarovich *et al* 1989) and is thought to be connected with the possibility of a consistent definition of the continuum limit for the system. In the present case the spectrum will be analytic if ν_c and ν_s are rational and only special values of N are allowed. To be definite, if

$$\nu_c = p_c/q_c \qquad \nu_s = p_s/q_s \tag{3.1}$$

with p_c and q_c (p_s and q_s) being relative prime integers, then only

$$N = qN' \tag{3.2}$$

values are allowed for N, where N' is an integer and q is the least integer dividable by both q_c and q_s. If (3.1) and (3.2) are met, the ground-state values N_c and N_s are

$$N_{c0} = p_c(q/q_c)N' \qquad N_{s0} = p_s(q/q_s)N' \tag{3.3}$$

and in the excited states

$$\Delta N_c = N_c - \nu_c N \qquad \Delta N_s = N_s - \nu_s N \tag{3.4}$$

are integers. Whether D_c and D_s are integers or half-odd integers depends on the parity of the numbers N_c and N_s. Due to the restrictions (2.2)

$$\begin{aligned} D_c &= (N_c + N_s + 1)/2 \qquad (\text{mod } 1) \\ D_s &= N_c/2 \qquad (\text{mod } 1). \end{aligned} \tag{3.5}$$

In the ground state both D are zero only if N_{c0} is even and N_{s0} is odd (otherwise at least one of the two D is $\pm\frac{1}{2}$, i.e. the ground state is degenerate). This imposes a restriction on the numbers p_c, p_s, q/q_c, q/q_s and N': p_s, q/q_s and N' should be odd, while one of p_c and q/q_c should be even. If these requirements are met, the finite-size corrections to the ground-state energy are

$$E_0 - N\varepsilon_\infty = -\frac{\pi v_c}{6N} - \frac{\pi v_s}{6N} \tag{3.6}$$

just as it would be in the case of two independent conformal fields. These fields are, however, not independent as all ΔN_c, ΔN_s, D_c and D_s appear multiplied by both Fermi velocities:

$$\begin{aligned} E - E_0 = {}& \frac{2\pi v_c}{N}\left(\frac{(\xi_{22}\Delta N_c - \xi_{21}\Delta N_s)^2}{4(\det \xi)^2} + (\xi_{11}D_c + \xi_{12}D_s)^2 + n_c^+ + n_c^-\right) \\ & + \frac{2\pi v_s}{N}\left(\frac{(\xi_{12}\Delta N_c - \xi_{11}\Delta N_s)^2}{4(\det \xi)^2} + (\xi_{21}D_c + \xi_{22}D_s)^2 + n_s^+ + n_s^-\right). \end{aligned} \tag{3.7}$$

Another interesting feature is that D_c and D_s are not independent of ΔN_c and ΔN_s, as even if the parameters are such that in the ground state $\Delta N_c = \Delta N_s = D_c = D_s = 0$, in the excited states due to (3.5)

$$\begin{aligned} D_c &= (\Delta N_c + \Delta N_s)/2 \qquad (\text{mod } 1) \\ D_s &= \Delta N_c/2 \qquad (\text{mod } 1). \end{aligned} \tag{3.8}$$

Examining (2.50) analogous conclusions can be drawn for the case of the attractive chain.

An important special case is when $U > 0$, $h = 0$, i.e. the ground state is non-magnetic ($\nu_s = \nu_c/2$). From our formulae we can get this case by taking the $\lambda_0 \to \infty$ limit. This can be done by solving the equations for the λ-dependent quantities at $\lambda \gg 1$ with Wiener–Hopf techniques and then taking the $\lambda_0 \to \infty$ limit. As a result one obtains that

$$\lim_{\lambda_0 \to \infty} \boldsymbol{\xi} = \begin{pmatrix} \xi & \xi/2 \\ 0 & 1/\sqrt{2} \end{pmatrix} \tag{3.9}$$

where

$$\xi = \xi(\sin k_0) \tag{3.10}$$

with $\xi(x)$ solving the equation

$$\begin{aligned} \xi(x) &= 1 + \frac{1}{2\pi}\int_{-\sin k_0}^{\sin k_0} \bar{K}(x - x')\xi(x')\,\mathrm{d}x' \\ \bar{K}(x) &= \int_{-\infty}^{\infty} \frac{\exp(-|\omega|u)}{2\cosh \omega u} \exp(\mathrm{i}\omega x)\,\mathrm{d}\omega. \end{aligned} \tag{3.11}$$

In this case (2.44) and (2.45) simplify to

$$E = N\varepsilon_\infty + \frac{2\pi v_c}{N}\left(\frac{(\Delta N_c)^2}{4\xi^2} + \xi^2(D_c + D_s/2)^2 - \tfrac{1}{12} + n_c^+ + n_c^-\right)$$
$$+\frac{2\pi v_s}{N}(\tfrac{1}{2}S^2 + \tfrac{1}{2}D_s^2 - \tfrac{1}{12} + n_s^+ + n_s^-) \tag{3.12}$$

and

$$P = \frac{2\pi}{N} N_{c0}(D_c + D_s/2) + \frac{2\pi}{N}[\Delta N_c(D_c + D_s/2) + n_c^+ + n_c^-]$$
$$+\frac{2\pi}{N}(-SD_s + n_s^+ - n_s^-) \tag{3.13}$$

with

$$S = N_c/2 - N_s. \tag{3.14}$$

It is also interesting to give the Fermi velocities in this limit:

$$2\pi v_c = \varepsilon_c'(k_0)/\rho_c(k_0)$$
$$2\pi v_s = \left(\int_{-k_0}^{k_0} \exp\left(\frac{\pi}{2u}\sin k\right)\varepsilon_c'(k)\right)\left(\int_{-k_0}^{k_0} \exp\left(\frac{\pi}{2u}\sin k\right)\rho_c(k)\right)^{-1}. \tag{3.15}$$

Here $\rho_c(k)$ and $\varepsilon_c'(k)$ satisfy the equations

$$\rho_c(k) = \frac{1}{2\pi} + \frac{1}{2\pi}\cos(k)\int_{-k_0}^{k_0} \bar{K}(\sin k - \sin k')\rho_c(k') \tag{3.16}$$

and

$$\varepsilon_c'(k) = 2\sin k + \frac{1}{2\pi}\cos(k)\int_{-k_0}^{k_0} \bar{K}(\sin k - \sin k')\varepsilon_c'(k'). \tag{3.17}$$

For the negative U case the $\lambda_0 \to \infty$ limit corresponds to the half-filled band $\nu_f + 2\nu_b = 1$.

We have to comment also on the $k_0 \to \pi$ limit. For the positive U case this corresponds to the half-filled band. In a strictly half-filled band $N_c = N$, and there are no such charge excitations as described in this work since there is no place for the particles (for $N_c = N$ the only possible charge excitations are those with complex k, but those have a gap). It is possible to create such particle-hole excitations only if first N_c is decreased, i.e. a $\Delta N_c < 0$ is introduced. Even in this case, however, we do not get a contribution to the energy as $v_c = 0$ if $k_0 = \pi$ (indicating that the spectrum is quadratic, i.e. the excitation energy $O(1/N^2)$. If U is negative, the $k_0 = \pi$ limit corresponds to the zero magnetisation, i.e. to states with no 'free' particles, only bound pairs. It is possible to break up pairs to create free particles and this does not cost large amounts of energy since the magnetic field and chemical potential are such that both the bottom of the band for the free particles and the Fermi points of the sea of bound pairs are at zero. The fact that $v_c = 0$ indicates that the spectrum of the free particles is quadratic, as it should be at the bottom of a band.

An important case is when $v_c = v_s$: in this point the model is conformally invariant with $c = 2$. In the Hubbard model (1.4) there are three parameters U, μ and h, or equivalently U, ν_c and ν_s. In principle one can define through the equation

$$v_c(U, \nu_c, \nu_s) = v_s(U, \nu_c, \nu_c) \tag{3.18}$$

a function $U(\nu_c, \nu_s)$ which gives the value of U where (3.18) is satisfied at the given ν_c and ν_s. Not all (ν_c, ν_s) pairs define such a U, but for a certain part of the $0<2\nu_s \leqslant \nu_c<1$ parameter space such a U can be found. (An example for this is the $\nu_s = \nu_c/2$ case, where for $\nu_c \to 0$ $(k_0 \to 0)$, $v_s/v_c \to 0$, while for $\nu_c \to 1$ $(k_0 \to \pi)$, $v_s/v_c \to \infty$ at any U (see (3.15)), so there is a ν_c to any U where $v_c = v_s$. This means that there is a whole range in $0<\nu_c<1$ where, with $\nu_s = \nu_c/2$, (3.18) can be satisfied.) In those points, where ν_c and ν_s are rational and satisfy the requirements discussed in the first paragraphs of this section, the model has a conformally invariant continuum limit with $c=2$, provided U is chosen according to (3.18). In this case the scaling indices of the primary operators are

$$\begin{aligned} x(\Delta N_c, \Delta N_s, D_c, D_s) &= \left(\frac{(\xi_{22}\Delta N_c - \xi_{21}\Delta N_s)^2}{4(\det \xi)^2} + (\xi_{11}D_c + \xi_{12}D_s)^2\right) \\ &\quad + \left(\frac{(\xi_{12}\Delta N_c - \xi_{11}\Delta N_s)^2}{4(\det \xi)^2} + (\xi_{21}D_c + \xi_{22}D_s)^2\right) \qquad (3.19) \\ s(\Delta N_c, \Delta N_s, D_c, D_s) &= \Delta N_c D_c + \Delta N_s D_s \end{aligned}$$

with ΔN_c, ΔN_s being integers, and D_c, D_s satisfying (3.8). It is worth noting that (3.19) is a generalisation of the Gaussian form: with the notation

$$\begin{pmatrix}\Delta N_c \\ \Delta N_s\end{pmatrix} = \boldsymbol{\Delta N} \qquad \begin{pmatrix}D_c \\ D_s\end{pmatrix} = \boldsymbol{D} \qquad \boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{\xi} = \mathbf{X} \tag{3.20}$$

we have

$$\begin{aligned} x(\boldsymbol{\Delta N}, \boldsymbol{D}) &= \tfrac{1}{4}\boldsymbol{\Delta N}^{\mathrm{T}}\mathbf{X}^{-1}\boldsymbol{\Delta N} + \boldsymbol{D}^{\mathrm{T}}\mathbf{X}\boldsymbol{D} \\ s(\boldsymbol{\Delta N}, \boldsymbol{D}) &= \boldsymbol{\Delta N}^{\mathrm{T}}\boldsymbol{D}. \end{aligned} \tag{3.21}$$

In addition to the above operators, there is a class of operators with $\boldsymbol{\Delta N} = \boldsymbol{D} = 0$ and

$$x = n_c^+ + n_c^- + n_s^+ + n_s^- \qquad s = n_c^+ - n_c^- + n_s^+ - n_s^- . \tag{3.22}$$

It is remarkable that four of them are marginal ($x=2$; $s=0$). Although we cannot read out the scaling indices from the spectrum directly if $v_c \neq v_s$, we expect that some of these operators are marginal even if $v_c \neq v_s$. The reason for this is that in the most general 1D model of spin-$\frac{1}{2}$ fermions there are several coupling constants and the model is critical in a whole region of a four-dimensional parameter space (for a review see Sólyom 1979). Thus there must be a set of marginal operators which govern the motion of the Hamiltonian in this parameter space. The Hubbard model is one special line parametrised by U in the critical region of the more general model, and the operator which, by adding it to the Hamiltonian, changes the value of U can be constructed readily:

$$\sum_i \left(n_{i\uparrow}n_{i\downarrow} + \frac{\partial \mu}{\partial u}(n_{i\uparrow} + n_{i\downarrow}) - \frac{1}{2}\frac{\partial h}{\partial u}(n_{i\uparrow} - n_{i\downarrow}) \right) \tag{3.23}$$

where $\partial\mu/\partial u$ and $\partial h/\partial u$ are partial derivatives at fixed ν_c and ν_s. This operator does not change the complete integrability of the model. Nevertheless—if U was such that $v_c = v_s$—it drives out the system from the conformally invariant point. The operators driving the system off the Hubbard line are not present explicitly in the Hubbard Hamiltonian. Nevertheless they should also be marginal.

Acknowledgments

I am very grateful to Drs V E Korepin, N M Bogoliubov and N Yu Reshetikhin and to Professor J Sólyom for extremely interesting and valuable discussions. This work was partially supported by DFG under contract Seho-158/8-1.

References

Affleck I 1986 *Phys. Rev. Lett.* **56** 746
Belavin A A, Polyakov A M and Zamolodchikov A B 1984 *J. Stat. Phys.* **34** 763
Berkovich A and Murthy G 1988 *J. Phys. A: Math. Gen.* **21** L381
Blöte H W J, Cardy J L and Nightingale M P 1986 *Phys. Rev. Lett.* **56** 742
Bogoliubov N M, Izergin A G and Reshetikhin N Yu 1987 *J. Phys. A: Math. Gen.* **20** 5631
Cardy J L 1986a *Nucl. Phys.* B **270** [FS16] 186
—— 1986b *Phase Transitions and Critical Phenomena* vol 11, ed C Domb and J L Lebowitz (New York: Academic)
de Vega H J and Woynarovich F 1985 *Nucl. Phys.* B **251** [FS13] 439
Friedan D, Qiu Z and Shenker S 1984 *Phys. Rev. Lett.* **52** 1575
Korepin V E 1979 *Teor. Mat. Fiz.* **41** 169
Korepin V E, Bogoliubov N M and Reshetikhin N Yu 1988 private communication
Lieb E M and Wu F Y 1968 *Phys. Rev. Lett.* **20** 1445
Sólyom J 1979 *Adv. Phys.* **28** 201
Suzuki J 1988 *Preprint* Tokyo
Woynarovich F 1982a *J. Phys. C: Solid State Phys.* **15** 85
—— 1982b *J. Phys. C: Solid State Phys.* **15** 97
—— 1983a *J. Phys. C: Solid State Phys.* **16** 5293
—— 1983b *J. Phys. C: Solid State Phys.* **16** 6593
Woynarovich F and Eckle H-P 1987a *J. Phys. A: Math. Gen.* **20** L97
—— 1987b *J. Phys. A: Math. Gen.* **20** L443
Woynarovich F, Eckle H-P and Truong T T 1989 *J. Phys. A: Math. Gen.* **22** 4027